

Overconnections and the energy-tensors of gauge and gravitational fields

Daniel Canarutto

Dipartimento di Matematica e Informatica “U. Dini”,

Via S. Marta 3, 50139 Firenze, Italia

email: daniel.canarutto@unifi.it

<http://www.dma.unifi.it/~canarutto>

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Abstract

A geometric construction for obtaining a prolongation of a connection to a connection of a bundle of connections is presented. This determines a natural extension of the notion of canonical energy-tensor which suits gauge and gravitational fields, and shares the main properties of the energy-tensor of a matter field in the jet space formulation of Lagrangian field theory, in particular with regards to symmetries of the Poincaré-Cartan form. Accordingly, the joint energy-tensor for interacting matter and gauge fields turns out to be a natural geometric object, whose definition needs no auxiliary structures. Various topics related to energy-tensors, symmetries and the Einstein equations in a theory with interacting matter, gauge and gravitational fields can be viewed under a clarifying light. Finally, the symmetry determined by the “Komar superpotential” is expressed as a symmetry of the gravitational Poincaré-Cartan form.

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Introduction

The so-called “canonical energy-tensor” relates “basic” vector fields, that is vector fields on the source manifold, to infinitesimal symmetries of the field theory under consideration and currents associated with them. While it is usually appreciated that the standard expression $\ell \delta_b^a - \phi_{,b}^i \partial_i^a \ell$ has no geometric meaning on non-trivial bundles, this issue is dealt with in various ways and with different formalisms, sometimes with *ad hoc* prescriptions. On the whole we may say that issues related to energy-tensors, including the stress-energy tensor in General Relativity, continue to generate much interest and discussions in the literature [1, 2, 3, 4, 8, 12, 13, 21, 29, 33, 35].

It can be argued that the natural setting for a thorough clarification of such matters is provided by the geometric formulation of Lagrangian field theory on jet spaces and, in particular, by generalizations of the Noether theorem expressed in terms of symmetries of the Poincaré-Cartan form [10, 15, 11, 16, 22, 24, 25, 26, 27, 30, 37, 38, 39, 40]. In that context, a previous paper by M. Modugno and myself [6] offered a natural extension of the notion of canonical energy-tensor to the non-trivial bundle case, on the basis of an earlier suggestion by Hermann [19].

The basic notions related to the Poincaré-Cartan form are reviewed here in §2.1; the above said construction of the energy-tensor, which uses a fixed background connection of the involved bundle, is reviewed in §2.2. In §2.4 we discuss two basic examples and, in those contexts, the canonical energy-tensor is compared to the stress-energy tensor appearing in the right-hand side of the Einstein equations when the considered field is coupled to gravity.

The extension of the above said approach to the case when the considered field is a connection poses additional problems. First we should note that a general connection of a bundle $E \rightarrow M$ cannot be characterized as a section of a finite-dimensional bundle over the base manifold M . However a smooth algebraic structure of the fibers does select a special family of connections, which can be regarded as the sections of a natural finite-dimensional bundle $C \rightarrow M$. This idea was first introduced by Garcia [14] in the context of principal bundles, and then generalized and used by Modugno and others [9, 32, 5, 7, 20], sometimes with the designation “systems of connections”. One interesting aspect of that formalism is that the bundle $C \rightarrow M$ inherits a fibered algebraic structure, which in turn determines a new system of connections of it, called “overconnections” (or “connections over connections”).

The needed notions related to systems of connections and overconnections are introduced in §1.1, §1.2 and §1.3. In particular we consider linear connections of vector bundles. In this context we explicitly describe the bundle $C \rightarrow M$ and its sub-bundles determined by possible further fiber structures, as well as the induced systems of overconnections. Moreover we show how, via natural geometric constructions, a linear connection can be lifted to an overconnection with the aid of a linear connection of the base manifold.

This lift of connections to overconnections, presented here for the first time as far as I know, naturally yields the wanted definitions of energy-tensors for gauge fields (§3.1). Though the proposed construction essentially arises by analogy with the canonical energy-tensor of a matter field described in §2.2, it is fully justified *a posteriori* by its properties. Indeed we find that the same relations among basic vector fields and symmetries of the Poincaré-Cartan form still hold. Furthermore the total energy-tensor for a theory of interacting matter and gauge fields on a gravitational background arises now very naturally, and is independent of any auxiliary structures. This total object (not the single pieces) turns out to be divergence-free on-shell, as one would expect.

The further extension of these ideas to a theory of coupled matter, gauge and gravitational

fields is still quite natural (§3.2). Indeed the definition of energy-tensor for the gravitational field, in the so-called “metric-affine” approach, essentially follows from the same procedure and fulfills the expected properties in relation to the Poincaré-Cartan form. In this enhanced geometric context we can view under a different light issues and results discussed with special emphasis by Padmanabhan [34], in particular about the relation among basic vector fields, energy-tensors, symmetries and the Einstein equations.

In §3.3 we examine the off-shell symmetry of the gravitational field determined by any vector field on the spacetime manifold [23, 34]. This turns out to be indeed related to a symmetry of the Poincaré-Cartan form of the gravitational field, obtained via a certain natural lift which is different, and somewhat more complicated, than the horizontal lift used in the definition of the energy-tensor.

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1 Connections and overconnections

1.1 Prolongations of general connections

According to a certain line of thought, the basic ideas related to the notion of a connection are best expressed in a general context in which no fiber structure is assumed in the bundle under consideration. A possible algebraic fiber structure enters the picture at a later step, selecting a special family of connections. Indeed, a connection of an arbitrary finite-dimensional smooth fibered manifold $\mathbf{E} \rightarrow \mathbf{M}$ is defined to be a (smooth) section $\kappa : \mathbf{E} \rightarrow \mathbf{JE}$ (we denote by \mathbf{T} , \mathbf{V} and $\mathbf{J} \equiv \mathbf{J}_1$ the tangent, vertical and first-jet prolongation functors). We recall that $\mathbf{p_E} : \mathbf{JE} \rightarrow \mathbf{E}$ is naturally an affine bundle, as

$$\mathbf{dl} : \mathbf{JE} \hookrightarrow \mathbf{T}^*\mathbf{M} \otimes_{\mathbf{E}} \mathbf{TE}$$

is the sub-bundle over \mathbf{E} which projects over the identity section $\mathbf{M} \rightarrow \mathbf{T}^*\mathbf{M} \otimes \mathbf{TM}$.

The tangent prolongation $\mathbf{T}\phi : \mathbf{TM} \rightarrow \mathbf{TE}$ of a local section $\phi : \mathbf{M} \rightarrow \mathbf{E}$ can be regarded as a section $\mathbf{M} \rightarrow \mathbf{T}^*\mathbf{M} \otimes_{\mathbf{E}} \mathbf{TE}$ and, since it projects over the identity of \mathbf{TM} , also as a section $\mathbf{j}\phi : \mathbf{M} \rightarrow \mathbf{JE}$ (the *first-jet prolongation* of ϕ). If (x^a, y^i) is a local fibered coordinate chart of \mathbf{E} then we denote by (x^a, y^i, y_a^i) the induced fibered chart of \mathbf{JE} , namely we have

$$y_a^i \circ \mathbf{j}\phi = \phi_{,a}^i \equiv \partial_a \phi^i .$$

The components of a connection κ are the functions

$$\kappa_a^i \equiv y_a^i \circ \kappa : \mathbf{E} \rightarrow \mathbb{R} .$$

Since κ can be regarded as a tangent-valued one-form on \mathbf{E} , its *curvature tensor* can be introduced, in terms of the Frölicher-Nijenhuis bracket, as the vertical-valued two-form

$$\rho := -\llbracket \kappa, \kappa \rrbracket = \rho_{ab}^i dx^a \wedge dx^b \otimes \partial y_i : \mathbf{E} \rightarrow \wedge^2 \mathbf{T}^*\mathbf{M} \otimes_{\mathbf{E}} \mathbf{VE}$$

where $\rho_{ab}^i = \kappa_{a,b}^i - \kappa_{b,a}^i + \partial_j \kappa_a^i \kappa_b^j - \partial_j \kappa_b^i \kappa_a^j$.

The *vertical projection* associated with κ is the vertical-valued 1-form

$$\omega := \mathbb{1}_{\mathbf{TE}} - \kappa : \mathbf{E} \rightarrow \mathbf{T}^*\mathbf{E} \otimes_{\mathbf{E}} \mathbf{VE} ,$$

which yields the *covariant derivative* of a section $\phi : \mathbf{M} \rightarrow \mathbf{E}$ as

$$\nabla\phi := j\phi_!\omega : \mathbf{M} \rightarrow \mathbf{T}^*\mathbf{M} \otimes_{\mathbf{E}} \mathbf{V}\mathbf{E} .$$

We are now interested in seeing under which conditions a connection κ of $\mathbf{E} \rightarrow \mathbf{M}$ can be lifted to a connection of $\mathbf{J}\mathbf{E} \rightarrow \mathbf{M}$. We start by noting that the first jet prolongation of $\kappa : \mathbf{E} \rightarrow \mathbf{J}\mathbf{E}$ is a morphism

$$\mathbf{J}\kappa : \mathbf{J}\mathbf{E} \rightarrow \mathbf{J}\mathbf{J}\mathbf{E} .$$

This is not a connection of $\mathbf{J}\mathbf{E} \rightarrow \mathbf{M}$, because it is not a section of $\mathbf{p}_{\mathbf{J}\mathbf{E}} : \mathbf{J}\mathbf{J}\mathbf{E} \rightarrow \mathbf{J}\mathbf{E}$ but rather a section of $\mathbf{J}\mathbf{p}_{\mathbf{E}} : \mathbf{J}\mathbf{J}\mathbf{E} \rightarrow \mathbf{J}\mathbf{E}$. We can express this argument in terms of local coordinates as follows. We denote the induced coordinate chart of $\mathbf{J}\mathbf{J}\mathbf{E}$ by

$$(x^a, y^i, \underline{y}_a^i; y_a^i, y_{ab}^i) , \quad y_{ab}^i \equiv (y_a^i)_b \neq y_{ba}^i ,$$

namely

$$y_a^i \circ \mathbf{p}_{\mathbf{J}\mathbf{E}} = \underline{y}_a^i , \quad y_a^i \circ \mathbf{J}\mathbf{p}_{\mathbf{E}} = y_a^i ,$$

and obtain

$$(x^a, y^i, \underline{y}_a^i; y_a^i, y_{ab}^i) \circ \mathbf{J}\kappa = (x^a, y^i, \kappa_a^i; y_a^i, \kappa_{a,b}^i + y_b^j \partial_j \kappa_a^i) .$$

Thus we could turn $\mathbf{J}\kappa$ into a connection of $\mathbf{J}\mathbf{E} \rightarrow \mathbf{M}$ if we availed of an involution of $\mathbf{J}\mathbf{J}\mathbf{E}$ exchanging \underline{y}_a^i and y_a^i . However this generalization of the involution of the double tangent space of a manifold requires some added structure, namely a symmetric linear connection Γ of $\mathbf{T}\mathbf{M} \rightarrow \mathbf{M}$. Indeed it has been proved by Modugno [31] that Γ determines a distinguished involution $s_\Gamma : \mathbf{J}\mathbf{J}\mathbf{E} \rightarrow \mathbf{J}\mathbf{J}\mathbf{E}$, with the coordinate expression (note the exchange $y_{ab}^i \rightarrow y_{ba}^i$)

$$(x^a, y^i, \underline{y}_a^i; y_a^i, y_{ab}^i) \circ s_\Gamma = (x^a, y^i, y_a^i; \underline{y}_a^i, y_{ba}^i + \Gamma_{ba}^c (\underline{y}_c^i - y_c^i)) .$$

Using the above cited result we now easily prove:

Theorem 1.1 *The composition*

$$\kappa' \equiv s_\Gamma \circ \mathbf{J}\kappa : \mathbf{J}\mathbf{E} \rightarrow \mathbf{J}\mathbf{J}\mathbf{E}$$

is a connection of $\mathbf{J}\mathbf{E} \rightarrow \mathbf{M}$ which is projectable over κ . Its coordinate expression turns out to be

$$(x^a, y^i, \underline{y}_a^i; y_a^i, y_{ab}^i) \circ \kappa' = (x^a, y^i, y_a^i; \kappa_a^i, \kappa_{b,a}^i + y_a^j \partial_j \kappa_b^i + \Gamma_{ba}^c (\kappa_c^i - y_c^i)) .$$

A somewhat more manageable form for the coordinate expression of κ' is

$$(\kappa'_a)^i = \kappa_a^i , \quad (\kappa'_a)^i_b = \kappa_{a,b}^i + y_b^j \partial_j \kappa_a^i + \Gamma_{ab}^c (\kappa_c^i - y_c^i) .$$

Moreover we remark that the projectability property of κ' can be expressed by the commutative diagram

$$\begin{array}{ccc} \mathbf{J}\mathbf{E} & \xrightarrow{\kappa'} & \mathbf{J}\mathbf{J}\mathbf{E} \\ \mathbf{p}_{\mathbf{E}} \downarrow & & \downarrow \mathbf{J}\mathbf{p}_{\mathbf{E}} \\ \mathbf{E} & \xrightarrow[\kappa]{} & \mathbf{J}\mathbf{E} \end{array}$$

1.2 Systems of linear connections and overconnections

In works by Modugno and others [9, 32] the term “overconnection” denotes a connection of a finite-dimensional “bundle of connections”, arising in the context of “system of connections”. A generic connection $\mathbf{E} \rightarrow \mathbf{J}\mathbf{E}$ cannot be characterized as a section of a finite-dimensional bundle over \mathbf{M} . In many relevant cases, however, one avails of a fibered algebraic structure of $\mathbf{E} \rightarrow \mathbf{M}$, selecting a special class of connections that can be seen as sections of some finite-dimensional bundle $\mathbf{C} \rightarrow \mathbf{M}$. More precisely, there is an “evaluation morphism”

$$\chi : \mathbf{E} \times_{\mathbf{M}} \mathbf{C} \rightarrow \mathbf{J}\mathbf{E} ,$$

such that every section $c : \mathbf{M} \rightarrow \mathbf{C}$ determines a special connection via the composition

$$\mathbf{E} \xrightarrow{(\text{id}, c \circ \mathbf{p}_M)} \mathbf{E} \times_{\mathbf{M}} \mathbf{C} \xrightarrow{\chi} \mathbf{J}\mathbf{E} ,$$

and, conversely, every special connection is obtained in this way. The map χ determines various structures on \mathbf{C} . The ensuing theory of system of connections (and, more generally, of systems of sections of 2-fibered bundles) has been studied in various contexts.

In the literature, the notion of bundle of connections has been mainly exploited in relation to principal connections in gauge field theories (e.g. see Janyška [20] and the references therein). The bundle of linear connections of a vector bundle provides an alternative, viable point of view, which, as far as I know, has not yet been thoroughly explored.

Let $\mathbf{E} \rightarrow \mathbf{M}$ be a vector bundle. Then $\mathbf{J}\mathbf{E} \rightarrow \mathbf{M}$ turns out to be a vector bundle too,¹ as the jet prolongation functor \mathbf{J} naturally lifts the algebraic fiber structure (the zero section and the vector space operations). Accordingly, we say that a connection $\kappa : \mathbf{E} \rightarrow \mathbf{J}\mathbf{E}$ is *linear* if it is a linear morphism over \mathbf{M} . In that case we can regard it as a section $\mathbf{M} \rightarrow \mathbf{J}\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^*$ projecting over the identity of \mathbf{E} . In other words, we can regard any linear connection of \mathbf{E} as a section $\mathbf{M} \rightarrow \mathbf{C}$ where

$$\mathbf{C} \subset \mathbf{J}\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^* ,$$

called the *bundle of linear connections of $\mathbf{E} \rightarrow \mathbf{M}$* , is the affine subbundle over \mathbf{M} which projects over the identity section $\mathbb{1}_{\mathbf{E}} : \mathbf{M} \rightarrow \mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^*$. The associated vector bundle is²

$$\mathbf{DC} = \mathbf{D}\mathbf{J}\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^* = \mathbf{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^* \equiv \mathbf{T}^*\mathbf{M} \otimes_{\mathbf{M}} \text{End } \mathbf{E} .$$

We now assume that the chosen fiber coordinates (y^i) are linear. Then we obtain induced coordinates (x^a, y^i_j, y^i_{aj}) on $\mathbf{J}\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^*$, so that \mathbf{C} is the submanifold locally characterized by the constraint $y^i_j = \delta^i_j$. If $c : \mathbf{M} \rightarrow \mathbf{C}$ is a section then the induced linear connection³ κ is locally characterized by the components $\kappa^i_a = \kappa^i_{aj} y^j$ with $\kappa^i_{aj} \equiv y^i_{aj} \circ c$.

Similar arguments hold for an affine bundle $\mathbf{F} \rightarrow \mathbf{M}$, namely we define the *affine connections* of it⁴ as the affine morphisms $\mathbf{F} \rightarrow \mathbf{J}\mathbf{F}$ which project over the identity of \mathbf{F} . Now, since the above introduced bundle $\mathbf{C} \rightarrow \mathbf{M}$ is affine, it has in turn a distinguished system of connections, called *the natural system of overconnections of \mathbf{E}* . Using theorem 1.1 we can now show that a distinguished overconnection naturally arises from objects that are available in a standard gauge field theory. First, by a coordinate computation one easily proves:

¹While $\mathbf{J}\mathbf{E} \rightarrow \mathbf{E}$ is still an affine bundle.

²In fact $\mathbf{V}\mathbf{E} \cong \mathbf{E} \times_{\mathbf{M}} \mathbf{E}$ because $\mathbf{E} \rightarrow \mathbf{M}$ is a vector bundle.

³In the present context it's usually safe to dispense with the explicit use of the evaluation morphism χ , so as a rule we identify $c : \mathbf{M} \rightarrow \mathbf{C}$ with $\kappa = \chi \circ (\text{id}, c \circ \mathbf{p}_M)$.

⁴The term “affine connection” here is not to be intended in the same sense as in many physics texts, where it essentially relates to the fiber structure of $\mathbf{J}\mathbf{E} \rightarrow \mathbf{E}$.

Lemma 1.1 *Let κ be a linear connection of $\mathbf{E} \rightarrow \mathbf{M}$ and Γ a linear connection of $\mathbf{T}\mathbf{M} \rightarrow \mathbf{M}$. Then $\kappa' \equiv s_\Gamma \circ J\kappa : J\mathbf{E} \rightarrow J\mathbf{E}$ is a linear connection of $J\mathbf{E} \rightarrow \mathbf{M}$, whose components are*

$$\begin{cases} (\kappa'_a)^i = \kappa_{aj}^i y^j, \\ (\kappa'_a)^i_b = \partial_b \kappa_{aj}^i y^j + y_b^j \kappa_{aj}^i + \Gamma_{ab}^c (\kappa_{cj}^i y^j - y_c^i). \end{cases}$$

Moreover standard arguments about induced connections of tensor product bundles yield:

Lemma 1.2 *Let κ and Γ be as in 1.1, and let κ^* be the “dual” linear connection of $\mathbf{E}^* \rightarrow \mathbf{M}$. Then $\kappa^\uparrow \equiv \kappa' \otimes \kappa^*$ is a linear connection of $J\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^* \rightarrow \mathbf{M}$, with coefficients*

$$\begin{cases} (\kappa_a^\uparrow)^i_j = -\kappa_{aj}^h y_h^i + \kappa_{ah}^i y_j^h, \\ (\kappa_a^\uparrow)^i_{bj} = \partial_b \kappa_{aj}^i y_j^h - \kappa_{aj}^h y_{bh}^i + y_{bj}^h \kappa_{ah}^i + \Gamma_{ab}^c (\kappa_{ch}^i y_j^h - y_{cj}^i). \end{cases}$$

Now from the observation that

$$VC \subset V(J\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^*) = (J\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^*) \times_{\mathbf{M}} (J\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^*)$$

is the subbundle which projects over $\mathbb{1} \times 0$ we find:

Theorem 1.2 *The connection κ^\uparrow of lemma 1.2 is reducible to an affine connection of the subbundle $\mathbf{C} \rightarrow \mathbf{M}$, whose coefficients turn out to be*

$$(\kappa_a^\uparrow)^i_{bj} = \partial_b \kappa_{aj}^i - \kappa_{aj}^h y_{bh}^i + y_{bj}^h \kappa_{ah}^i + \Gamma_{ab}^c (\kappa_{cj}^i - y_{cj}^i).$$

Summarizing, κ and Γ together determine an overconnection, that is an affine connection of $\mathbf{C} \rightarrow \mathbf{M}$, which we'll still denote as κ^\uparrow . In particular we may express the covariant derivative of κ itself with respect to κ^\uparrow , getting

$$\begin{aligned} \nabla_a \kappa_{bj}^i &= \partial_a \kappa_{bj}^i - \partial_b \kappa_{aj}^i + \kappa_{bh}^i \kappa_{aj}^h - \kappa_{ah}^i \kappa_{bj}^h = \\ &= -\rho_{abj}^i. \end{aligned}$$

Note that the contribution of the spacetime connection Γ disappears in the above expression.

1.3 Systems of gauge fields

In most situations of interest the vector bundle $\mathbf{E} \rightarrow \mathbf{M}$ is endowed with a richer fiber structure. The linear connections which preserve that structure, let's call them *gauge fields*, constitute a *subsystem*; namely, they can be characterized as sections of an affine sub-bundle of the bundle $\mathbf{C} \rightarrow \mathbf{M}$ introduced in §1.2.

The bundle of all linear endomorphisms of \mathbf{E} is $\text{End } \mathbf{E} \equiv \mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^* \rightarrow \mathbf{M}$. Its fibers are endowed with a natural Lie algebra structure given by the ordinary commutator. We denote by $\text{Aut } \mathbf{E}$ its sub-bundle of all invertible endomorphisms; this is a *group bundle*, and $\text{End } \mathbf{E}$ is its “Lie-algebra bundle”. Let now $\mathbf{G} \subset \text{Aut } \mathbf{E}$ be the sub-bundle of all automorphisms which preserve the assigned fiber structure of \mathbf{E} . Its Lie-algebra bundle is a sub-bundle $\mathcal{L} \subset \text{End } \mathbf{E}$, and the bundle of gauge fields is easily recognized as the affine sub-bundle $\mathbf{C}_G \subset \mathbf{C}$ over \mathbf{M} whose associated vector bundle is

$$D\mathbf{C}_G = T^*\mathbf{M} \otimes_{\mathbf{M}} \mathcal{L} \subset T^*\mathbf{M} \otimes_{\mathbf{M}} \text{End } \mathbf{E}.$$

Moreover, the curvature tensor of any $\kappa : \mathbf{M} \rightarrow \mathbf{C}_G$ can be regarded as a section

$$\rho : \mathbf{M} \rightarrow \wedge^2 T^* \mathbf{M} \otimes_{\mathbf{M}} \mathfrak{L} .$$

If one chooses a special frame of \mathbf{E} , coordinate calculations are essentially the same as in the familiar principal bundle formalism. Let's consider the common case in which the fiber is complex with fiber dimension n , and is endowed with a Hermitian metric: then \mathfrak{L} is the bundle of all *anti-Hermitian* endomorphisms ($A^\dagger = -A$). As a real vector bundle of fiber dimension $2n^2$, $\text{End } \mathbf{E}$ is endowed with the distinguished symmetric bilinear form $(A, B) \mapsto \Re \text{Tr}(A \circ B)$, whose signature turns out to be (n^2, n^2) .

Now the assigned Hermitian structure of \mathbf{E} also yields the Hermitian structure of $\text{End } \mathbf{E}$ given by $(A, B) \mapsto \text{Tr}(A^\dagger \circ B)$, and the real splitting $\text{End } \mathbf{E} = \mathfrak{L} \oplus_{\mathbf{M}} i \mathfrak{L}$ (any endomorphism can be uniquely written as the sum of anti-Hermitian and Hermitian terms). The restrictions of this Hermitian product to \mathfrak{L} and $i \mathfrak{L}$ turn out to be real positive (Euclidean), while the restrictions of the real scalar product have opposite signatures.

If (y_i) is an orthonormal frame of \mathbf{E} then the matrix of a section $\mathbf{M} \rightarrow \mathfrak{L}$ is anti-Hermitian. In particular, one can always find an orthonormal frame (l_i) of \mathfrak{L} related to (y_i) by the relations $l_i = l_{ij}^i y_i \otimes y^j$, where the matrices (l_{ij}^i) are *constant*. We then obtain the constant coefficients (*structure constants*)

$$c_{JH}^I \equiv \langle l^I, [l_J, l_H] \rangle ,$$

where (l^I) is the dual frame. Indices can be lowered and raised via the above said positive metric of \mathfrak{L} .

Accordingly, the coordinate expressions of a gauge field and that of its curvature can be written as

$$\kappa = dx^a \otimes (\partial x_a + \kappa_a^I l_I) , \quad \rho = \rho_{ab}^I dx^a \wedge dx^b \otimes l_I ,$$

with $\rho_{ab}^I = \kappa_{a,b}^I - \kappa_{b,a}^I + c_{JH}^I \kappa_a^J \kappa_b^H$. (In order to compare with the usual physics literature write $\kappa_a^I \equiv i q A_a^I$ and $\rho_{ab}^I \equiv i q F_{ab}^I$ with $q \in \mathbb{R}$.)

The *dual connection* κ^* and its curvature ρ^* can be viewed as valued in the dual Lie algebra bundle $\mathfrak{L}^* \subset \text{End } \mathbf{E}^* \equiv \mathbf{E}^* \otimes \mathbf{E}$, and are related to κ and ρ by *minus transposition*: in terms of fiber indices of \mathbf{E} we have

$$\kappa_{aj}^{*i} = -\kappa_{aj}^i , \quad \rho_{abj}^{*i} = -\kappa_{abj}^i .$$

In terms of Lie algebra indices we'll write their coordinate expressions as κ_{aI}^* and ρ_{abI}^* ; the operations $\kappa_a^I \mapsto \kappa_{aI}^*$ and $\rho_{ab}^I \mapsto \rho_{abI}^*$ can also be regarded as *index lowering* with respect to the above said Hermitian structure of $\text{End } \mathbf{E}$.

Recalling the results of §1.2 we now make the main point of this section, which is easily proved by inspecting the general expression of the overconnection κ^\dagger determined by κ with the aid of a suitable connection on the base manifold.

Theorem 1.3 *If we avail of a linear torsion-free connection Γ of $T\mathbf{M} \rightarrow \mathbf{M}$, then any gauge field κ determines a connection κ^\dagger of $\mathbf{C} \rightarrow \mathbf{M}$ which turns out to be reducible to a connection of $\mathbf{C}_G \rightarrow \mathbf{M}$, with the coordinate expression*

$$(\kappa_a^\dagger)_b^I = \kappa_{a,b}^I + c_{JH}^I \kappa_a^J y_b^H + \Gamma_{ab}^c (\kappa_c^I - y_c^I) .$$

As in the “unconstrained” case, the covariant derivative of κ with respect to κ^\dagger is independent of Γ and has the coordinate expression

$$\nabla_a \kappa_b^I = -\rho_{ab}^I .$$

The above result is related to the “Utiyama theorem” and its generalizations [20], which we’ll comment about further in §3.1.

2 Energy-tensor in Lagrangian field theories

2.1 Poincaré-Cartan form and currents

We recall a few basic notions in Lagrangian field theory. A *1-st order Lagrangian density* on a fibered manifold $\mathbf{E} \rightarrow \mathbf{M}$ is defined to be a morphism $\mathcal{L} : \mathbf{J}\mathbf{E} \rightarrow \wedge^m \mathbf{T}^* \mathbf{M}$ over \mathbf{M} , where $m \equiv \dim \mathbf{M}$. We write its coordinate expression as $\mathcal{L} = \ell d^m x$, with $d^m x \equiv dx^1 \wedge \cdots \wedge dx^m$. The associated *Euler-Lagrange operator*

$$\mathcal{E} : \mathbf{J}_2 \mathbf{E} \rightarrow \wedge^m \mathbf{T}^* \mathbf{M} \otimes_{\mathbf{E}} \mathbf{V}^* \mathbf{E}$$

has the coordinate expression⁵ $\mathcal{E}_i = \partial_i \ell - d_a \partial_i^a \ell$; critical sections $\phi : \mathbf{M} \rightarrow \mathbf{E}$ are characterized by the condition $\mathcal{E} \circ j_2 \phi = 0$.

As $\mathbf{J}\mathbf{E} \rightarrow \mathbf{E}$ is an affine bundle and its associated vector bundle is $\mathbf{D}\mathbf{J}\mathbf{E} = \mathbf{T}^* \mathbf{M} \otimes_{\mathbf{E}} \mathbf{V}\mathbf{E}$, the *fiber derivative* of \mathcal{L} has a well-defined meaning as a morphism

$$\mathbf{D}\mathcal{L} : \mathbf{J}\mathbf{E} \rightarrow (\mathbf{D}\mathbf{J}\mathbf{E})^* \otimes_{\mathbf{E}} \wedge^m \mathbf{T}^* \mathbf{M} = \mathbf{T}\mathbf{M} \otimes_{\mathbf{E}} \mathbf{V}^* \mathbf{E} \otimes_{\mathbf{E}} \wedge^m \mathbf{T}^* \mathbf{M}.$$

We can transform this object via some natural operations: we perform an obvious contraction and anti-symmetrization, and use the transpose of the *contact 1-form*⁶ $\vartheta : \mathbf{T}\mathbf{J}\mathbf{E} \rightarrow \mathbf{V}\mathbf{E}$. We end up with an m -form

$$\mathcal{P} = \mathcal{P}_i^a \vartheta^i \wedge dx_a \equiv \partial_i^a \ell (dy^i - y_b^i dx^b) \wedge dx_a : \mathbf{J}\mathbf{E} \rightarrow \wedge^m \mathbf{T}^* \mathbf{J}\mathbf{E},$$

which by analogy with mechanics is sometimes called “momentum”. Since we have natural inclusions $\mathbf{T}^* \mathbf{M} \subset \mathbf{T}^* \mathbf{E} \subset \mathbf{T}^* \mathbf{J}\mathbf{E}$, the *Poincaré-Cartan form* $\mathcal{C} \equiv \mathcal{L} + \mathcal{P}$ is a well-defined m -form on $\mathbf{J}\mathbf{E}$, too.

In the present context we deal with symmetries of the Poincaré-Cartan form rather than symmetries of the action functional (the latter is the most common way in which these matters are formulated in the physics literature). A *symmetry of \mathcal{C}* is a vector field $Z : \mathbf{J}\mathbf{E} \rightarrow \mathbf{T}\mathbf{J}\mathbf{E}$ such that the Lie derivative $L_Z \mathcal{C}$ vanishes along all critical sections,⁷ that is $j\phi^* L_Z \mathcal{C} = 0$.

The above definition can be refined via the following observations. Let $Y : \mathbf{J}\mathbf{E} \rightarrow \mathbf{T}\mathbf{E}$ be a morphism over \mathbf{E} . By noting that for any section ϕ one has $j\phi^* \vartheta^i = 0$, it is not difficult to check that the forms $j\phi^* i_Z \mathcal{C}$, $j\phi^* i_Z d\mathcal{C}$ and $j\phi^* L_Z \mathcal{C}$ are independent of the choice of a vector field $Z : \mathbf{J} \rightarrow \mathbf{T}\mathbf{J}\mathbf{E}$ such that $p_E \circ Z = Y$. Accordingly we say that Y is a symmetry of \mathcal{C} if $j\phi^* L_Z \mathcal{C} = 0$ for any such extension Z and for any critical section ϕ .

A *conserved current* is defined to be an $m-1$ -form $\mathcal{J} : \mathbf{J}\mathbf{E} \rightarrow \wedge^{m-1} \mathbf{T}^* \mathbf{J}\mathbf{E}$ such that the m -form $j\phi^* d\mathcal{J}$ on \mathbf{M} vanishes for any critical section ϕ . It can be proved that the condition that ϕ be critical can be equivalently expressed as $(j\phi)^*(i_Z d\mathcal{C}) = 0$ for any vector field Z ; then one immediately proves the following generalized version of the Noether theorem:

⁵If $f : \mathbf{J}\mathbf{E} \rightarrow \mathbb{R}$ then the functions $d_a f = \partial_a f + y_a^i \partial_i f + y_{ab}^i \partial_i^a f : \mathbf{J}_2 \mathbf{E} \rightarrow \mathbb{R}$ are the components of its *horizontal differential*, see e.g. Saunders [36].

⁶This is the natural morphism over \mathbf{E} with coordinate expression $\vartheta^i \equiv dy^i \circ \vartheta = dy^i - y_a^i dx^a$ [30].

⁷If α is any q -form on $\mathbf{J}\mathbf{E}$ then for any section $\phi : \mathbf{M} \rightarrow \mathbf{E}$ the pull-back $j\phi^* \alpha$ is a q -form on \mathbf{M} . In particular one has $dj\phi^* \alpha = j\phi^* d\alpha$.

Theorem 2.1

- If $Y : \mathbf{J}\mathbf{E} \rightarrow \mathbf{T}\mathbf{E}$ is a symmetry of \mathcal{C} then $i_Y \mathcal{C} : \mathbf{J}\mathbf{E} \rightarrow \wedge^{m-1} \mathbf{T}^* \mathbf{J}\mathbf{E}$ is a conserved current.
- If there exists an $m-1$ -form $\varphi : \mathbf{J}\mathbf{E} \rightarrow \wedge^{m-1} \mathbf{T}^* \mathbf{J}\mathbf{E}$ such that $j\phi^* \mathbf{L}_Z \mathcal{C} = j\phi^* d\varphi$, then, more generally, $i_Y \mathcal{C} - \varphi$ is a conserved current. This condition is locally equivalent to $j\phi^* \mathbf{L}_Z d\mathcal{C} = 0$.

For any $Y : \mathbf{J}\mathbf{E} \rightarrow \mathbf{T}\mathbf{E}$ a simple calculation yields

$$\begin{aligned} j\phi^*(i_Y \mathcal{C}) &= j\phi^*(i_Y \mathcal{L} + i_{Y-Y} \mathbf{d}\mathcal{P}) = \\ &= (\ell Y^a + \mathcal{P}_i^a (Y^i - Y^b \phi_{,b}^i)) dx_a . \end{aligned}$$

From this we see that there is a possible situation in which one easily finds a symmetry Y yielding a given current \mathcal{J} , that is when $\mathcal{J} = i_X \mathcal{L} + i_W \mathcal{P}$ with $X : \mathbf{J}\mathbf{E} \rightarrow \mathbf{T}\mathbf{M}$, $W : \mathbf{J}\mathbf{E} \rightarrow \mathbf{V}\mathbf{E}$. It's easy to check that in that case one such Y is obtained by setting

$$Y = X \lrcorner \mathbf{d} + W = X^a \partial_{x_a} + (X^a y_a^i + W^i) \partial_{y_i} ,$$

where $X = X^a \partial_{x_a}$, $W = W^i \partial_{y_i}$. We'll actually use this procedure in §3.3.

Remark. In the physics literature the above topics are usually expressed in terms of “covariant divergence” rather than of exterior differential. The relation between these two formalisms, which we'll further discuss in §2.3, is also called the “replacement principle” [20].

2.2 Canonical energy-tensor in non-trivial bundles

We are still in the context of a 1-st order Lagrangian field theory on a fibered manifold $\mathbf{E} \rightarrow \mathbf{M}$, but now we also assume a (provisionally fixed) connection $\kappa : \mathbf{E} \rightarrow \mathbf{J}\mathbf{E}$. We can then define the *canonical energy-tensor* as the morphism $\mathcal{U} : \mathbf{J}\mathbf{E} \rightarrow \wedge^{m-1} \mathbf{T}^* \mathbf{M} \otimes \mathbf{T}^* \mathbf{M}$ over \mathbf{M} which has the coordinate expression

$$\mathcal{U} = (\ell \delta_b^a - (y_b^i - \kappa_b^i) \partial_i^a \ell) dx_a \otimes dx^b .$$

The importance of this object lies in the possibility to consider certain symmetries which are generated by vector fields on the base manifold. Actually for any section $\phi : \mathbf{M} \rightarrow \mathbf{E}$ and for any vector field $X : \mathbf{M} \rightarrow \mathbf{T}\mathbf{M}$ we have

$$(\mathcal{U} \circ j\phi) \lrcorner X = j\phi^*(i_Y \mathcal{C}) ,$$

where $Y \equiv X \lrcorner \kappa : \mathbf{E} \rightarrow \mathbf{T}\mathbf{E}$ is the so-called *horizontal prolongation* of X . Thus, assuming that Y turns out to be a symmetry of \mathcal{C} , the conserved current evaluated through a critical section ϕ is $(\mathcal{U} \circ j\phi) \lrcorner X : \mathbf{M} \rightarrow \wedge^3 \mathbf{T}^* \mathbf{M}$.

We briefly review the geometric construction of \mathcal{U} [6]. The covariant derivative operator associated with κ can be regarded as a morphism $\nabla : \mathbf{J}\mathbf{E} \rightarrow \mathbf{T}^* \mathbf{M} \otimes_{\mathbf{E}} \mathbf{V}\mathbf{E}$. By performing suitable contractions of the tensor product $D\mathcal{L} \otimes \nabla$ we obtain the morphism

$$\langle D\mathcal{L} \otimes \nabla \rangle = (y_a^i - \kappa_a^i) \partial_i^b \ell dx^a \otimes dx_b : \mathbf{J}\mathbf{E} \rightarrow \wedge^{m-1} \mathbf{T}^* \mathbf{M} \otimes_{\mathbf{M}} \mathbf{T}^* \mathbf{M} .$$

Moreover we note that there is a natural inclusion $\iota : \wedge^m \mathbf{T}^* \mathbf{M} \hookrightarrow \wedge^{m-1} \mathbf{T}^* \mathbf{M} \otimes_{\mathbf{M}} \mathbf{T}^* \mathbf{M}$, characterized by $i_X \alpha = (\iota \alpha) \lrcorner X$, so that eventually we set $\mathcal{U} := \iota \mathcal{L} - \langle D\mathcal{L} \otimes \nabla \rangle$.

In order to compare the above \mathcal{U} with the stress-energy tensor \mathcal{T} of standard General Relativity we note that the latter is valued into $\mathbf{TM} \otimes_M \mathbf{TM} \otimes_M \wedge^4 \mathbf{T}^* \mathbf{M}$; indeed, when the matter Lagrangian density \mathcal{L} does not depend on the derivatives of the metric g , then \mathcal{T} is just the fiber derivative of \mathcal{L} with respect to g . By a contraction and index moving via g we obtain a morphism

$$\mathcal{T} = \mathcal{T}^a_b dx_a \otimes dx^b : \mathbf{JE} \rightarrow \wedge^3 \mathbf{T}^* \mathbf{M} \otimes_M \mathbf{T}^* \mathbf{M} .$$

So \mathcal{U} and \mathcal{T} are valued in the same bundles, and it's easy to see that their “physical dimensions” match too. While they not necessarily coincide for an arbitrary Lagrangian density, in physically relevant cases they turn out to be either equal or closely related.⁸

2.3 Remarks about the “replacement principle”

If the base manifold of a field theory is Lorentz spacetime, with positively oriented unit volume form η , then a current $\mathcal{J} = \mathcal{J}^a dx_a$ can be treated as the vector field $J : \mathbf{M} \rightarrow \mathbf{TM}$ characterized by $\mathcal{J} = i_J \eta$, that is $J = J^a \partial_{x_a}$ with $J^a \sqrt{|g|} = \mathcal{J}^a$. Accordingly, the exterior differential $d\mathcal{J}$ can be replaced by the covariant “divergence” $\nabla_a J^a$, as it's not difficult to check that if the spacetime connection is torsion-free then actually $d\mathcal{J} = \nabla_a J^a \sqrt{|g|} d^4x$, so that the two formalisms could be easily merged.

We can extend the above procedure as follows. We assume that $\mathbf{E} \rightarrow \mathbf{M}$ is a vector bundle (hence $\mathbf{VE} \cong \mathbf{E} \times_M \mathbf{E}$). We consider a linear connection $\kappa : \mathbf{E} \rightarrow \mathbf{JE}$ besides the spacetime connection Γ , and a section

$$\xi = \xi^{ai} dx_a \otimes \partial y_i : \mathbf{M} \rightarrow \wedge^3 \mathbf{T}^* \mathbf{M} \otimes \mathbf{VE} .$$

Then we have the Frölicher-Nijenhuis bracket

$$[\kappa, \xi] = (\partial_a \xi^{ai} - \kappa_{aj}^i \xi^{aj}) d^4x \otimes \partial y_i : \mathbf{M} \rightarrow \wedge^4 \mathbf{T}^* \mathbf{M} \otimes \mathbf{VE} .$$

Moreover we have the covariant derivative of ξ with respect to (Γ, κ) , with the coordinate expression

$$\nabla \xi = (\partial_c \xi^{ai} - \Gamma_{cb}^a \xi^{bi} + \Gamma_{cb}^b \xi^{ai} - \kappa_{cj}^i \xi^{aj}) dx^c \otimes dx_a \otimes \partial y_i .$$

As this is valued in $\mathbf{T}^* \mathbf{M} \otimes_M \wedge^3 \mathbf{T}^* \mathbf{M} \otimes_{\mathbf{E}} \mathbf{VE}$ we can antisymmetrize the horizontal factors, thus obtaining the *covariant divergence*

$$\begin{aligned} \nabla \cdot \xi &= (\partial_a \xi^{ai} - \kappa_{aj}^i \xi^{aj} + \tau_a \xi^{ai}) d^4x \otimes \partial y_i = \\ &= [\kappa, \xi] + \tau \wedge \xi , \end{aligned}$$

where $\tau_a \equiv \Gamma_{ac}^c - \Gamma_{ca}^c$ is the *torsion* 1-form.

Via natural algebraic operations involving the volume form η we also introduce

$$\check{\xi} = \check{\xi}^{ai} \partial x_a \otimes \partial y_i : \mathbf{M} \rightarrow \mathbf{TM} \otimes_M \mathbf{E} , \quad \check{\xi}^{ai} \equiv \frac{1}{\sqrt{|g|}} \xi^{ai} ,$$

and by a straightforward computation we find

$$\nabla \cdot \xi = \nabla \cdot \check{\xi} \otimes \eta ,$$

⁸See e.g. Gotay-Marsden [17] for a discussion about relations between these two types of objects.

where

$$\nabla \cdot \check{\xi} \equiv \nabla_a \check{\xi}^{ai} \partial y_i = \frac{1}{\sqrt{|g|}} (\partial_a \xi^{ai} - \kappa_{aj}^i \xi^{aj} + \tau_a \xi^{ai}) \partial y_i .$$

For handling energy-tensors we just set $\mathbf{E} \equiv T^* \mathbf{M}$, and get

$$\nabla \cdot \mathcal{U} = \nabla \cdot \check{\mathcal{U}} \otimes \eta = \nabla_a \check{\mathcal{U}}^a_b \sqrt{|g|} \, dx^b \otimes d^4 x , \quad \check{\mathcal{U}}^a_b \equiv \frac{1}{\sqrt{|g|}} \mathcal{U}^a_b .$$

2.4 Basic examples

In concrete examples, particularly when the theory under consideration has several sectors, dropping the rigorous distinction between fiber coordinates and field components can be notationally convenient, and usually won't generate confusion if some care is used. Accordingly we'll also write \mathcal{T} and \mathcal{U} for $\mathcal{T} \circ j\phi$ and $\mathcal{U} \circ j\phi$. The expressions of the energy tensors and their covariant divergences found in the two following examples are essentially standard results, though we stress that, differently from usual presentations, a field and its conjugate are seen here as independent sections of mutually dual bundles. The most important point about these examples is their association with the energy-tensor of a gauge field (§3.1).

2.4.1 Charged spin-zero field (non-abelian case)

In this example $\mathbf{F} \rightarrow \mathbf{M}$ is a vector bundle endowed with a linear connection κ , and (\mathbf{M}, g) is Einstein spacetime. Provisionally, both κ and g are considered as fixed background structures. The “configuration bundle” is $\mathbf{E} \equiv \mathbf{F} \oplus_{\mathbf{M}} \mathbf{F}^*$, so that a field is actually a couple of sections, $\phi : \mathbf{M} \rightarrow \mathbf{F}$ and $\bar{\phi} : \mathbf{M} \rightarrow \mathbf{F}^*$. In standard presentations the fibers are complex and ϕ and $\bar{\phi}$ are regarded as mutually adjoint via some Hermitian structure preserved by κ , but that specification is not needed here.

The Lagrangian density $\mathcal{L} = \ell \, d^4 x$ is given by

$$\ell = \frac{1}{2} (g^{ab} \nabla_a \bar{\phi}_i \nabla_b \phi^i - m^2 \bar{\phi}_i \phi^i) \sqrt{|g|} ,$$

where $\nabla_a \phi^i = \partial_a \phi^i - \kappa_{aj}^i \phi^j$, $\nabla_a \bar{\phi}_i = \partial_a \bar{\phi}_i + \kappa_{ai}^j \bar{\phi}_j$, and m is a constant mass.

We have

$$\mathcal{P}_i^a \equiv \partial_i^a \ell = \frac{1}{2} g^{ac} \nabla_c \bar{\phi}_i , \quad \mathcal{P}^{ai} \equiv \partial^{ai} \ell = \frac{1}{2} g^{ac} \nabla_c \phi^i ,$$

whence we obtain

$$\begin{aligned} \mathcal{U}_b^a &= \ell \delta_b^a - \mathcal{P}_i^a \nabla_b \phi^i - \nabla_b \bar{\phi}_i \mathcal{P}^{ai} = \\ &= \frac{1}{2} (g^{cd} \nabla_c \bar{\phi}_i \nabla_d \phi^i \delta_b^a - g^{ac} (\nabla_c \bar{\phi}_i \nabla_b \phi^i + \nabla_b \bar{\phi}_i \nabla_c \phi^i) - m^2 \bar{\phi}_i \phi^i \delta_b^a) \sqrt{|g|} , \end{aligned}$$

Next, using $\partial \sqrt{|g|} / \partial g^{ab} = -\frac{1}{2} g_{ab} \sqrt{|g|}$, we obtain

$$\begin{aligned} \mathcal{T}_{ab} &= \frac{1}{4} \nabla_{\{a} \bar{\phi}_i \nabla_{b\}} \phi^i \sqrt{|g|} - \frac{1}{2} g_{ab} \ell = \\ &= \frac{1}{4} (\nabla_a \bar{\phi}_i \nabla_b \phi^i + \nabla_b \bar{\phi}_i \nabla_a \phi^i) \sqrt{|g|} - \frac{1}{4} g_{ab} (g^{cd} \nabla_c \bar{\phi}_i \nabla_d \phi^i - m^2 \bar{\phi}_i \phi^i) \sqrt{|g|} , \end{aligned}$$

where braces delimiting indices denote symmetrization (without dividing by the appropriate factorial). By a simple further calculation we then obtain

$$g_{ac} \mathcal{U}_b^c = -2 \mathcal{T}_{ab} .$$

We evaluate $\nabla \cdot \mathcal{U}$ on-shell, that is by taking the field equations into account. These can be expressed in terms of the Frölicher-Nijenhuis bracket in the form

$$[\![\kappa, * \nabla \phi]\!] + m^2 \phi \eta = 0, \quad [\![\kappa, * \nabla \bar{\phi}]\!] + m^2 \bar{\phi} \eta = 0,$$

where $*$ is the standard Hodge isomorphism, or in coordinates as

$$\begin{aligned} \partial_a (g^{ab} \sqrt{|g|} \nabla_b \phi^i) - g^{ab} \sqrt{|g|} \kappa_{aj}^i \nabla_b \phi^j + m^2 \sqrt{|g|} \phi^i &= 0, \\ \partial_a (g^{ab} \sqrt{|g|} \nabla_b \bar{\phi}_i) + g^{ab} \sqrt{|g|} \kappa_{ai}^j \nabla_b \bar{\phi}_j + m^2 \sqrt{|g|} \bar{\phi}_i &= 0. \end{aligned}$$

A computation then yields the on-shell expression

$$\nabla_a \check{\mathcal{U}}^a_b = \frac{1}{2} g^{ac} \rho_{abj}^i (\bar{\phi}_i \nabla_c \phi^j - \nabla_c \bar{\phi}_i \phi^j).$$

2.4.2 Dirac field

Let $\mathbf{W} \hookrightarrow \mathbf{M}$ be the bundle of Dirac spinors over Einstein spacetime. In this context we indicate by $\gamma : \mathbf{TM} \rightarrow \text{End } \mathbf{W}$ the Dirac map, while the background *spinor connection* splits as

$$\mathbb{F}_{a\beta}^\alpha = i A_a \delta_{\beta}^\alpha + \frac{1}{4} \Gamma_a^{\lambda\mu} (\gamma_\lambda \gamma_\mu)^\alpha_{\beta},$$

where A is real and represents the e.m. field and Γ is the spacetime connection, whose components are expressed here in an orthonormal frame (θ_λ) of $\mathbf{TM} \hookrightarrow \mathbf{M}$ (a “tetrad”). As in the previous example we consider two independent fields $\psi : \mathbf{M} \rightarrow \mathbf{W}$ and $\bar{\psi} : \mathbf{M} \rightarrow \mathbf{W}^*$, which are usually regarded as mutually related by the *Dirac adjunction*, determined by a Hermitian metric with signature $(++--)$.

We set

$$\begin{aligned} \ell &= \left(\frac{i}{2} (\bar{\psi} \not{\nabla} \psi - \not{\nabla} \bar{\psi} \psi) - m \bar{\psi} \psi \right) \sqrt{|g|} \equiv \\ &\equiv \left(\frac{i}{2} g^{ab} (\bar{\psi}_\alpha \gamma_a^\alpha{}_\beta \nabla_b \psi^\beta - \nabla_a \bar{\psi}_\alpha \gamma_b^\alpha{}_\beta \psi^\beta) - m \bar{\psi}_\alpha \psi^\alpha \right) \sqrt{|g|}, \end{aligned}$$

where $\nabla_a \psi^\alpha = \partial_a \bar{\psi}_\alpha - \mathbb{F}_{a\beta}^\alpha \psi^\beta$, $\nabla_a \bar{\psi}_\alpha = \partial_a \bar{\phi}_\alpha + \mathbb{F}_{a\alpha}^\beta \bar{\psi}_\beta$. Then

$$\begin{aligned} \mathcal{T}_{ab} &= \frac{\partial \ell}{\partial g^{ab}} = \frac{i}{4} (\bar{\psi}_\alpha \gamma_{\{a}^\alpha{}_\beta \nabla_{b\}} \psi^\beta - \nabla_{\{a} \bar{\psi}_\alpha \gamma_{b\}}^\alpha{}_\beta \psi^\beta) \sqrt{|g|} - \frac{1}{2} \ell g_{ab} \equiv \\ &\equiv \frac{i}{4} (\bar{\psi} \gamma_{\{a} \nabla_{b\}} \psi - \nabla_{\{a} \bar{\psi} \gamma_{b\}} \psi) \sqrt{|g|} - \frac{1}{2} \ell g_{ab}. \end{aligned}$$

Moreover we have

$$\mathcal{P}_\alpha^a \equiv \partial_\alpha^a \ell = \frac{i}{2} (\bar{\psi} \gamma^a)_\alpha \sqrt{|g|}, \quad \mathcal{P}^{a\alpha} \equiv \partial^{a\alpha} \ell = -\frac{i}{2} (\gamma^a \psi)^\alpha \sqrt{|g|},$$

whence

$$\begin{aligned} \mathcal{U}^a_b &= -\frac{i}{2} (\bar{\psi} \gamma^a \nabla_b \psi - \nabla_b \bar{\psi} \gamma^a \psi) + \ell \delta^a_b, \\ \mathcal{U}_{ab} &= -\frac{i}{2} (\bar{\psi} \gamma_a \nabla_b \psi - \nabla_b \bar{\psi} \gamma_a \psi) + \ell g_{ab} \Rightarrow \\ \Rightarrow \frac{1}{2} \mathcal{U}_{\{ab\}} &= -\frac{i}{2} (\bar{\psi} \gamma_{\{a} \nabla_{b\}} \psi - \nabla_{\{a} \bar{\psi} \gamma_{b\}} \psi) \sqrt{|g|} + \ell g_{ab} = \\ &= -2 \mathcal{T}_{ab}. \end{aligned}$$

Next we evaluate the divergence of \mathcal{T} on-shell, that is by taking the Dirac equations

$$\gamma^a \nabla_a \psi = -i m \psi, \quad \nabla_a \bar{\psi} \gamma^a = i m \bar{\psi},$$

into account (for simplicity we are considering the case when the torsion of the spacetime connection vanishes). A not-so-short computation then yields

$$\nabla_a \check{\mathcal{T}}^a_b = \frac{i}{2} \rho_{ab} \bar{\psi} \gamma^a \psi = \frac{1}{2} F_{ab} \bar{\psi} \gamma^a \psi.$$

3 Energy-tensor of a connection

Henceforth, as in the previous examples, we'll consistently use the notational simplification of dropping the distinction between fiber coordinates and field components. In the literature, the adjective “formal” is sometimes attached to maps defined on jet bundles in a setting of this type; so for example one writes “formal curvature” [24] and the like.

3.1 Energy-tensor of a gauge field

We now use results of previous sections in order to propose a definition of energy-tensor of a gauge field. It's worthwhile stressing how this object naturally associates itself with the energy-tensors of matter fields (§2.4).

Let M be a spacetime with fixed background metric. In a theory of a gauge field $\kappa : M \rightarrow C_G$ (§1.3) one uses the natural Lagrangian $\mathcal{L}_{\text{gauge}} = \ell_{\text{gauge}} d^4x$ with

$$\ell_{\text{gauge}} = -\frac{1}{4} g^{ac} g^{bd} \rho_{abI}^* \rho_{cd}^I \sqrt{|g|}.$$

Since we also have the background Riemannian connection associated with the spacetime metric, a gauge field yields an overconnection κ^\uparrow which is reducible to a connection of $C_G \rightarrow M$: in its expression given in §1.3 we just replace the generic Γ with the Levi-Civita connection. Thus we are naturally led use κ^\uparrow in order to extend the construction of the canonical energy-tensor offered in §2.2: we insert the covariant derivative of κ with respect to κ^\uparrow in the place that there is occupied by the covariant derivative of the field with respect to the background connection. While this is not exactly the same procedure, it will be justified by the properties obeyed by the considered object.

In relation to the Utiyama theorem [20] we note that viewing ρ as the covariant derivative of κ puts gauge and matter fields on a more similar footing: the derivatives of both fields enter the total Lagrangian only through the covariant derivatives.

The momentum components for κ and its covariant derivative are

$$\mathcal{P}^{ac}_I \equiv \frac{\partial \ell_{\text{gauge}}}{\partial \kappa_{c,a}^I} = \rho^{ac}_I, \quad \nabla_b \kappa_c^I = -\rho_{bc}^I,$$

whence by applying the above sketched procedure we get the canonical energy-tensor $\mathcal{U}_{\text{gauge}}$ with components

$$\mathcal{U}^a_b = \ell \delta^a_b - \mathcal{P}^{ac}_I \nabla_b \kappa_c^I = \left(\rho^{ac}_I \rho_{bc}^I - \frac{1}{4} \rho^{cd}_I \rho_{cd}^I \delta^a_b \right) \sqrt{|g|}.$$

When $\kappa \equiv iA$ where A the electromagnetic 4-potential, $\mathcal{U}_{\text{gauge}}$ is seen to coincide with the Maxwell stress-energy tensor.

We also note that, analogously to the examples in §2.4, we find

$$-\frac{1}{2}\mathcal{U}_{ab} = \mathcal{T}_{ab} \equiv \partial\ell/\partial g^{ab} .$$

Let now $X : \mathbf{M} \rightarrow \mathbf{TM}$ and denote by

$$Y \equiv X_{\lrcorner}\kappa^{\dagger} = X^c (\partial x_c + (\kappa_c^{\dagger})_{bj}^i \partial y_i^{bj}) : \mathbf{C}_G \rightarrow \mathbf{TC}_G$$

its horizontal lift through κ^{\dagger} . We use coordinates $(x^a, y_a^I, y_{a,b}^I)$ on \mathbf{JC}_G , and write

$$\mathcal{P}_{\text{gauge}} = \mathcal{P}^{ac}_I \vartheta^I_c \wedge dx_a \equiv \mathcal{P}^{ac}_I (dy_c^I - y_{c,b}^I dx^b) \wedge dx_a .$$

Then by a coordinate computation it's not difficult to prove:

Theorem 3.1 *For each section $\kappa : \mathbf{M} \rightarrow \mathbf{C}_G$ we have*

$$(\mathcal{U}_{\text{gauge}} \circ j\kappa)_{\lrcorner} X = j\kappa^*(i_Y \mathcal{C}_{\text{gauge}}) , \quad \mathcal{C}_{\text{gauge}} \equiv \mathcal{L}_{\text{gauge}} + \mathcal{P}_{\text{gauge}} .$$

The above theorem constitutes a first justification for our calling $\mathcal{U}_{\text{gauge}}$ the canonical energy-tensor of the gauge field. We can offer a further justification, related to the vanishing of the divergence of energy tensors. Indeed it can be shown by a general naturality argument [28, 18] that the energy-tensor for a field theory on a gravitational background must be divergence-free. In order to check that this requirement is fulfilled we need to consider a theory of interacting matter and gauge fields

$$(\phi, \kappa) : \mathbf{M} \rightarrow \mathbf{E} \times_{\mathbf{M}} \mathbf{C}_G .$$

As a typical basic example, we first consider the charged boson field of §2.4.1 together with the appropriate gauge field. The Lagrangian density for this theory is assumed to be just the sum $\mathcal{L} = \mathcal{L}_{\phi} + \mathcal{L}_{\text{gauge}}$. Note that \mathcal{L}_{ϕ} depends on κ but not on its derivatives, while $\mathcal{L}_{\text{gauge}}$ is independent of ϕ . Then also the total momentum is the sum $\mathcal{P} = \mathcal{P}_{\phi} + \mathcal{P}_{\text{gauge}}$, and the total energy-tensor is the sum $\mathcal{U} = \mathcal{U}_{\phi} + \mathcal{U}_{\text{gauge}}$. The field equations for κ can be expressed in terms of the Frölicher-Nijenhuis bracket in the form

$$[\kappa, * \rho]_i^{aj} + \frac{1}{2} g^{ab} (\bar{\phi}_i \nabla_b \phi^j - \nabla_b \bar{\phi}_i \phi^j) = 0 ,$$

whence we get the on-shell divergence

$$\nabla \cdot (\check{\mathcal{U}}_{\text{gauge}})_b = \frac{1}{2} g^{ac} (\bar{\phi}_i \nabla_c \phi^j - \nabla_c \bar{\phi}_i \phi^j) \rho_{ba}^i .$$

Thus eventually, taking the replacement principle into account, we find

$$\nabla \cdot (\mathcal{T}_{\phi} + \mathcal{T}_{\text{gauge}}) = 0 .$$

Analogous computations yield the same result in the case of a theory of interacting fermion and gauge fields. Restricting our attention to the abelian case for simplicity, the field equations for κ and the on-shell divergence of $\mathcal{U}_{\text{gauge}}$ are now

$$[\kappa, * \rho]^a - i \bar{\psi} \gamma^a \psi = 0 , \quad \nabla_a \check{\mathcal{U}}^a_b[\kappa] = i \bar{\psi} \gamma^a \psi \rho_{ab} .$$

Then $\nabla \cdot (\mathcal{T}_{\psi} + \mathcal{T}_{\text{gauge}}) = 0$, where \mathcal{T}_{ψ} is related to \mathcal{U}_{ψ} by symmetrization (§2.4.2).

3.2 Energy-tensor of the gravitational field

A convenient Lagrangian formulation of gravity, discussed in the literature, is obtained by letting the spacetime metric g and the symmetric spacetime connection Γ be independent variables, and assuming the Lagrangian density to be $\mathcal{L}_{\text{grav}} = R \sqrt{|g|} \, d^4x$ where $R \equiv g^{ac} R_{abc}^b$ denotes the scalar curvature. The Euler-Lagrange equations turn out to be the Einstein equation for the g sector, and the metricity condition $\nabla[\Gamma]g = 0$ for the Γ sector.

Let $\mathbf{C}_\Gamma \rightarrow \mathbf{M}$ be the bundle of symmetric linear connections of $\mathbf{TM} \rightarrow \mathbf{M}$. Then any $\Gamma : \mathbf{M} \rightarrow \mathbf{C}_\Gamma$ determines an overconnection $\Gamma^\dagger : \mathbf{C}_\Gamma \rightarrow \mathbf{JC}_\Gamma$; we don't have to worry about an auxiliary connection on the base manifold since we avail of Γ itself. We can now proceed analogously to §3.1. The “covariant derivative” of Γ with respect to Γ^\dagger is $\nabla_a \Gamma_{bd}^c = -R_{abd}^c$, and since the Lagrangian is independent of the derivatives of the metric the momentum map $\mathcal{P}_{\text{grav}}$ has the simple coordinate expression

$$\begin{aligned} \mathcal{P}_{\text{grav}} &= \mathcal{P}_c^{abd} (dy_{bd}^c \wedge dx_a - y_{bd,a}^c \, d^m x) , \\ \mathcal{P}_c^{abd} &= (-g^{ad} \delta_c^b + g^{bd} \delta_c^a) \sqrt{|g|} . \end{aligned}$$

By the way, we remark that the above expression is frequent in physics. In particular, it is related to coefficients Q_{cd}^{ab} used e.g. by Padmanabhan [34] by $\mathcal{P}_{cd}^{ab} = 2 \sqrt{|g|} \, Q_{cd}^{ab}$.

We also remark that Krupka has studied energy-tensors in terms of variational forms [24]. In that approach the gravitational field is represented by the metric alone (a choice which makes certain details a little more involved).

The energy-tensor turns out to be essentially the Einstein tensor G , since we get

$$\begin{aligned} (\mathcal{U}_{\text{grav}})^a_b &= R \sqrt{|g|} \, \delta_b^a - \nabla_b \Gamma_{ed}^c \mathcal{P}_c^{aed} = (R \delta_b^a + R_{bed}^c (g^{ad} \delta_c^e - g^{ed} \delta_c^a)) \sqrt{|g|} = \\ &= -2 (R_b^a - \tfrac{1}{2} R \delta_b^a) \sqrt{|g|} \equiv -2 G_b^a \sqrt{|g|} . \end{aligned}$$

The horizontal Γ^\dagger -lift of $X : \mathbf{M} \rightarrow \mathbf{TM}$ is the vector field $X_J \Gamma^\dagger : \mathbf{C}_\Gamma \rightarrow \mathbf{TC}_\Gamma$. Remembering that $\nabla_a G_b^a = 0$, and letting $\mathcal{C}_{\text{grav}} = \mathcal{L}_{\text{grav}} + \mathcal{P}_{\text{grav}}$ be the Poincaré-Cartan form for this gravitational setting, a short computation also yields

$$j\Gamma^* d(i_{X_J \Gamma^\dagger} \mathcal{C}_{\text{grav}}) = \langle \mathcal{U}_{\text{grav}}, \nabla X \rangle = -2 G_b^a \nabla_b X^a \sqrt{|g|} \, d^4x .$$

Let's now consider a theory of interacting matter, gauge and gravitational fields, and the total energy-tensor $\mathcal{T} = \mathcal{T}_{\text{matter}} + \mathcal{T}_{\text{gauge}} + G = -\tfrac{1}{2} \mathcal{U}$. Then the condition that \mathcal{T} vanishes is equivalent to the Einstein equations

$$G = -(\mathcal{T}_{\text{matter}} + \mathcal{T}_{\text{gauge}}) .$$

The total bundle $\mathbf{Z} \rightarrow \mathbf{M}$ for this theory has a matter sector, a gauge sector, and a gravitational sector which in turn has a g sector and a Γ sector. Gathering the various constructions we readily realize that the couple (κ, Γ) , constituted by a gauge field and a spacetime connection, determines a prolongation of any basic vector field $X : \mathbf{M} \rightarrow \mathbf{TM}$ to a vector field $Y[X] : \mathbf{Z} \rightarrow \mathbf{TZ}$. Now, knowing that $\mathcal{T}_{\text{matter}} + \mathcal{T}_{\text{gauge}}$ is identically divergence-free on-shell, we can reformulate a known result [34] in terms of symmetries of the *total* Poincaré-Cartan form $\mathcal{C} = \mathcal{C}_{\text{grav}} + \mathcal{C}_{\text{matter}} + \mathcal{C}_{\text{gauge}}$:

Theorem 3.2 *The Einstein field equations follow from the requirement that $i_{Y[X]} \mathcal{C}$ be a conserved current for every basic vector field X .*

3.3 The off-shell symmetry associated with any vector field

If \mathbf{M} is an m -dimensional manifold then any exterior form $\varphi : \mathbf{M} \rightarrow \wedge^{m-2} \mathbf{T}^* \mathbf{M}$ trivially yields the conserved current $d\varphi$ for any field theory in which \mathbf{M} is the “source”. In this section \mathbf{M} is the spacetime manifold and the Riemannian connection is assumed to be torsion-free. Then for any 2-form φ we also consider the current $\frac{1}{2} d*\varphi$ ($*$ is the Hodge isomorphism). This can be expressed by a version of the replacement principle (§2.1) in terms of the covariant divergence, since we have

$$\frac{1}{2} d*\varphi = \nabla_a \varphi^{ab} \sqrt{|g|} dx_b .$$

Hence we can associate a current to any vector field $X : \mathbf{M} \rightarrow \mathbf{TM}$ by setting

$$\mathcal{J} := \frac{1}{2} d*[g^b(X)] , \quad g^b(X) \equiv g_{ac} X^c dx^a : \mathbf{M} \rightarrow \mathbf{T}^* \mathbf{M} .$$

Since the torsion is assumed to vanish we also have

$$\frac{1}{2} d*[g^b(X)]_{ab} = \frac{1}{2} \nabla_{[a} X_{b]} \sqrt{|g|} ,$$

which is known in the literature as the “Komar superpotential” [23, 34]. Now we can rewrite the current as

$$\mathcal{J} = \nabla_a \nabla^{[a} X^{b]} \sqrt{|g|} dx_b \equiv J^b \sqrt{|g|} dx_b ,$$

and its closeness can be expressed as $\nabla_b J^b = 0$. We remark that this is an *off-shell* symmetry, namely it holds independently of the field equation possibly obeyed by the gravitational field.

While this current is associated with a vector field on \mathbf{M} , it is not obtained as $\mathcal{U}_\Gamma X$, where $\mathcal{U} \equiv \mathcal{U}_{\text{grav}} = -2G$ (§3.2); namely if $\Gamma : \mathbf{M} \rightarrow \mathbf{C}_\Gamma$ is an arbitrary section, then the contraction of the horizontal lift $X_\Gamma \Gamma^\dagger : \mathbf{C}_\Gamma \rightarrow \mathbf{TC}_\Gamma$ with the Poincaré-Cartan form $\mathcal{C}_{\text{grav}}$ does not yield the conserved current \mathcal{J} . Hence it is natural to look for a different lift Y of X such that $\mathcal{J} \circ j\Gamma = j\Gamma^*(i_Y \mathcal{C}_{\text{grav}})$. We’ll use a procedure sketched in §2.1.

We first recall that the *Lie derivative of the connection Γ with respect to X* is the tensor field $L_X \Gamma : \mathbf{M} \rightarrow \mathbf{T}^* \mathbf{M} \otimes \mathbf{TM} \otimes \mathbf{T}^* \mathbf{M}$ characterized by [41]

$$L_X \Gamma_\Gamma Z = \nabla L_X Z - L_X \nabla Z$$

for any vector field $Z : \mathbf{M} \rightarrow \mathbf{TM}$. We obtain the coordinate expression

$$\begin{aligned} L_X \Gamma_{ac}^b &= -\partial_{ac} X^b + \partial_a X^d \Gamma_{dc}^b + \Gamma_{ad}^b \partial_c X^d - \Gamma_{ac}^d \partial_d X^b + X^d \partial_d \Gamma_{ac}^b = \\ &= -\nabla_a \nabla_c X^b - X^d R_{dac}^b , \end{aligned}$$

whence

$$J^b = \nabla_a \nabla^{[a} X^{b]} = -g^{ac} L_X \Gamma_{ac}^b + g^{ab} L_X \Gamma_{ac}^c + 2 R_a^b X^a .$$

The Poincaré-Cartan form for this case is $\mathcal{C}_{\text{grav}} \equiv \mathcal{L}_{\text{grav}} + \mathcal{P}_{\text{grav}}$, already written in §3.2. Then one immediately checks that the current’s components can be rewritten as

$$J^a = -\mathcal{P}_{ac}^{bd} L_X \Gamma_{bd}^c + 2 R_a^b X^b .$$

If now $Y = Y^a \partial_{x_a} + Y_{ac}^b \partial y_b^{ac} : \mathbf{JC}_\Gamma \rightarrow \mathbf{TC}_\Gamma$ is a morphism over \mathbf{C}_Γ , then we obtain

$$j\Gamma^*(i_Y \mathcal{C}_{\text{grav}}) = (R Y^a + \mathcal{P}_{ac}^{bd} (Y_{bc}^a - Y^d \partial_d \Gamma_{bc}^a)) \sqrt{|g|} dx_a .$$

The condition $j\Gamma^*(i_Y \mathcal{C}_{\text{grav}}) = \mathcal{J} \circ j\Gamma$ can be expressed as

$$R Y^a + \mathcal{P}_{ac}^{bd} (Y_{bd}^c - \Gamma_{bd,e}^c Y^e) = -\mathcal{P}_{ac}^{bd} L_X \Gamma_{bd}^c + 2 R_a^b X^b .$$

Since we are looking for *one* solution Y , the first obvious assumption is that Y projects over X , that is $Y^a = X^a$. Furthermore, by a straightforward computation we also get

$$\mathcal{P}_c^{abd} (R_{bde}^c X^e - R_{bd} X^c) = 2 R_b^a X^b - R X^a ,$$

so that eventually we are led to write the equation

$$\mathcal{P}_c^{abd} (Y_{bd}^c - X^e \partial_e \Gamma_{bd}^c + L_X \Gamma_{bd}^c - R_{bde}^c X^e + R_{bd} X^c) = 0$$

in which the unknowns are the components Y_{bd}^c . One solution is

$$Y_{bd}^c = X^e \partial_e \Gamma_{bd}^c - L_X \Gamma_{bd}^c + R_{bde}^c X^e - R_{bd} X^c .$$

Summarizing, and writing our result in a somewhat more precise form:

Theorem 3.3 *For any vector field $X : M \rightarrow TM$, the morphism*

$$Y = X^a \partial_{x_a} + Y_{bd}^c \partial y_c^{bd} : JC_\Gamma \rightarrow TC_\Gamma ,$$

$$Y_{bd}^c \circ j\Gamma = X^e \partial_e \Gamma_{bd}^c - L_X \Gamma_{bd}^c + R_{bde}^c[\Gamma] X^e - R_{bd}[\Gamma] X^c ,$$

yields the standard current $\mathcal{J}[X]$ associated with X as $i_Y C$.

Remark. In coordinate-free form we may express Y as

$$Y = X \lrcorner d - L_X \Gamma + \text{Riemann} \lrcorner X - \text{Ricci} \otimes X ,$$

where the last three terms are valued in $VC_\Gamma \cong C_\Gamma \times_M T^*M \otimes TM \otimes T^*M$.

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